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ON THE DETERMINATION OF ELLIPTIC ORBITS FROM THREE COMPLETE OBSERVATIONS.

By J. Willard Gibbs.

The determination of an orbit from three complete observations by the solution of the equations which represent elliptic motion presents so great difficulties in the general case, that in the first solution of the problem we must generally limit ourselves to the case in which the intervals between the observations are not very long. In this case we substitute some comparatively simple relations between the unknown quantities of the problem, which have an approximate validity for short intervals, for the less manageable relations which rigorously subsist between these quantities. A comparison of the approximate solution thus obtained with the exact laws of elliptic motion will always afford the means of a closer approximation, and by a repetition of this process we may arrive at any required degree of accuracy.

It is therefore a problem not without interest—it is, in fact, the natural point of departure in the study of the determination of orbits—to express in a manner combining as far as possible simplicity and accuracy the relations between three positions in an orbit separated by small or moderate intervals. The problem is not entirely determinate, for we may lay the greater stress upon simplicity or upon accuracy; we may seek the most simple relations which are sufficiently accurate to give us any approximation to an orbit, or we may seek the most exact expression of the real relations, which shall not be too complex to be serviceable.

DERIVATION OF THE FUNDAMENTAL EQUATION.

The following very simple considerations afford a vector equation, not very complex and quite amenable to analytical transformation, which expresses the relations between three positions in an orbit separated by small or moderate intervals, with an accuracy far exceeding that of the approximate relations generally used in the determination of orbits.

If we adopt such a unit of time that the acceleration due to the sun's action is unity at a unit's distance, and denote the vectors* drawn from the sun to the body in its three positions by

* Vectors, or *directed quantities*, will be represented in this paper by German capitals. The following notations will be used in connection with them.

The sign = denotes identity in direction as well as length.

The sign + denotes geometrical addition, or what is called composition in mechanics.

The sign — denotes reversal of direction, or composition after reversal.

The notation $\mathfrak{A}\cdot\mathfrak{B}$ denotes the product of the lengths of the vectors and the cosine of the angle which they include. It will be called the *direct product* of $\mathfrak{A}$ and $\mathfrak{B}$. If x, y, z are the rectangular components of $\mathfrak{A}$, and x', y', z' those of $\mathfrak{B}$,

$$\mathfrak{A}\cdot\mathfrak{B}=xx'+yy'+zz'.$$

$\mathfrak{A}\cdot\mathfrak{A}$ may be written $\mathfrak{A}^2$ and called the square of $\mathfrak{A}$.

The notation $\mathfrak{A}\times\mathfrak{B}$ will be used to denote a vector of which the length is the product of the lengths of $\mathfrak{A}$ and $\mathfrak{B}$ and the sine of the angle which they include. Its direction is perpendicular to $\mathfrak{A}$ and $\mathfrak{B}$, and on that side on which

$\mathfrak{R}_1$, $\mathfrak{R}_2$, $\mathfrak{R}_3$, and the lengths of these vectors (the heliocentric distances) by r_1, r_2, r_3, the accelerations corresponding to the three positions will be represented by $-\frac{\mathfrak{R}_1}{r_1^3}, -\frac{\mathfrak{R}_2}{r_2^3}, -\frac{\mathfrak{R}_3}{r_3^3}$. Now the motion between the positions considered may be expressed with a high degree of accuracy by an equation of the form

$$\mathfrak{R}=\mathfrak{A}+t\mathfrak{B}+t^2\mathfrak{C}+t^3\mathfrak{D}+t^4\mathfrak{E},$$

having five vector constants. The actual motion rigorously satisfies six conditions, viz., if we write τ_3 for the interval of time between the first and second positions, and τ_1 for that between the second and third, and set $t=0$ for the second position,

for $t=-\tau_3$,

$$\mathfrak{R}=\mathfrak{R}_1, \qquad \frac{d^2\mathfrak{R}}{dt^2}=-\frac{\mathfrak{R}_1}{r_1^3};$$

for $t=0$,

$$\mathfrak{R}=\mathfrak{R}_2, \qquad \frac{d^2\mathfrak{R}}{dt^2}=-\frac{\mathfrak{R}_2}{r_2^3};$$

for $t=\tau_1$,

$$\mathfrak{R}=\mathfrak{R}_3, \qquad \frac{d^2\mathfrak{R}}{dt^2}=-\frac{\mathfrak{R}_3}{r_3^3}.$$

We may therefore write with a high degree of approximation:

$$\mathfrak{R}_1=\mathfrak{A}-\tau_3\mathfrak{B}+\tau_3^2\mathfrak{C}-\tau_3^3\mathfrak{D}+\tau_3^4\mathfrak{E}$$

$$\mathfrak{R}_2=\mathfrak{A}$$

$$\mathfrak{R}_3=\mathfrak{A}+\tau_1\mathfrak{B}+\tau_1^2\mathfrak{C}+\tau_1^3\mathfrak{D}+\tau_1^4\mathfrak{E}$$

$$-\frac{\mathfrak{R}_1}{r_1^3}=2\mathfrak{C}-6\tau_3\mathfrak{D}+12\tau_3^2\mathfrak{E}$$

$$-\frac{\mathfrak{R}_2}{r_2^3}=2\mathfrak{C}$$

$$-\frac{\mathfrak{R}_3}{r_3^3}=2\mathfrak{C}+6\tau_1\mathfrak{D}+12\tau_1^2\mathfrak{E}.$$

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a rotation from $\mathfrak{A}$ to $\mathfrak{B}$ appears counter-clock-wise. It will be called the *skew product* of $\mathfrak{A}$ and $\mathfrak{B}$. If the rectangular components of $\mathfrak{A}$ and $\mathfrak{B}$ are x, y, z, and x', y', z', those of $\mathfrak{A}\times\mathfrak{B}$ will be

$$yz'-zy', \qquad zx'-xz', \qquad xy'-yx'.$$

The notation $(\mathfrak{A}\mathfrak{B}\mathfrak{C})$ denotes the volume of the parallelopiped of which three edges are obtained by laying off the vectors $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$ from any same point, which volume is to be taken positively or negatively, according as the vector $\mathfrak{C}$ falls on the side of the plane containing $\mathfrak{A}$ and $\mathfrak{B}$, on which a rotation from $\mathfrak{A}$ to $\mathfrak{B}$ appears counter-clock-wise, or on the other side. If the rectangular components of $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$ are x, y, z; x', y', z'; and x'', y'', z'',

$$(\mathfrak{A}\mathfrak{B}\mathfrak{C})=\begin{vmatrix} x & y & z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix}$$

It follows, from the above definitions, that for any vectors $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$

$$\mathfrak{A}\cdot\mathfrak{B}=\mathfrak{B}\cdot\mathfrak{A}, \qquad \mathfrak{A}\times\mathfrak{B}=-\mathfrak{B}\times\mathfrak{A}, \qquad (\mathfrak{A}\mathfrak{B}\mathfrak{C})=(\mathfrak{B}\mathfrak{C}\mathfrak{A})=(\mathfrak{C}\mathfrak{A}\mathfrak{B})=-(\mathfrak{A}\mathfrak{C}\mathfrak{B})=-(\mathfrak{C}\mathfrak{B}\mathfrak{A})=-(\mathfrak{B}\mathfrak{A}\mathfrak{C}),$$

and

$$(\mathfrak{A}\mathfrak{B}\mathfrak{C})=\mathfrak{A}\cdot(\mathfrak{B}\times\mathfrak{C})=\mathfrak{B}\cdot(\mathfrak{C}\times\mathfrak{A})=\mathfrak{C}\cdot(\mathfrak{A}\times\mathfrak{B});$$

also that $\mathfrak{A}\cdot\mathfrak{B}$, $\mathfrak{A}\times\mathfrak{B}$, are distributive functions of $\mathfrak{A}$ and $\mathfrak{B}$, and $(\mathfrak{A}\mathfrak{B}\mathfrak{C})$ a distributive function of $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$, for example, that if $\mathfrak{A}=\mathfrak{L}+\mathfrak{M}$,

$$\mathfrak{A}\cdot\mathfrak{B}=\mathfrak{L}\cdot\mathfrak{B}+\mathfrak{M}\cdot\mathfrak{B}, \qquad \mathfrak{A}\times\mathfrak{B}=\mathfrak{L}\times\mathfrak{B}+\mathfrak{M}\times\mathfrak{B}, \qquad (\mathfrak{A}\mathfrak{B}\mathfrak{C})=(\mathfrak{L}\mathfrak{B}\mathfrak{C})+(\mathfrak{M}\mathfrak{B}\mathfrak{C}),$$

and so for $\mathfrak{B}$ and $\mathfrak{C}$.

The notation $(\mathfrak{A}\mathfrak{B}\mathfrak{C})$ is identical with that of Lagrange in the *Mécanique Analytique*, except that there its use is limited to unit vectors. The signification of $\mathfrak{A}\times\mathfrak{B}$ is closely related to, but not identical with, that of the notation $[r_1r_2]$ commonly used to denote the double area of a triangle determined by two positions in an orbit.

From these six equations the five constants $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$ may be eliminated, leaving a single equation of the form

$$A_1\left(1+\frac{B_1}{r_1^3}\right)\mathfrak{R}_1-\left(1-\frac{B_2}{r_2^3}\right)\mathfrak{R}_2+A_3\left(1+\frac{B_3}{r_3^3}\right)\mathfrak{R}_3=0, \qquad (1)$$

where

$$A_1=\frac{\tau_1}{\tau_1+\tau_3} \qquad A_3=\frac{\tau_3}{\tau_1+\tau_3}$$

$$B_1=\tfrac{1}{12}(-\tau_1^2+\tau_1\tau_3+\tau_3^2) \qquad B_2=\tfrac{1}{12}(\tau_1^2+3\tau_1\tau_3+\tau_3^2) \qquad B_3=\tfrac{1}{12}(\tau_1^2+\tau_1\tau_3-\tau_3^2).$$

This we shall call our fundamental equation. In order to discuss its geometrical signification, let us set

$$n_1=A_1\left(1+\frac{B_1}{r_1^3}\right) \qquad n_2=\left(1-\frac{B_2}{r_2^3}\right) \qquad n_3=A_3\left(1+\frac{B_3}{r_3^3}\right), \qquad (2)$$

so that the equation will read

$$n_1\mathfrak{R}_1-n_2\mathfrak{R}_2+n_3\mathfrak{R}_3=0. \qquad (3)$$

This expresses that the vector $n_2\mathfrak{R}_2$ is the diagonal of a parallelogram of which $n_1\mathfrak{R}_1$ and $n_3\mathfrak{R}_3$ are sides. If we multiply by $\mathfrak{R}_3$ and by $\mathfrak{R}_1$, *in skew multiplication*, we get

$$n_1\mathfrak{R}_1\times\mathfrak{R}_3-n_2\mathfrak{R}_2\times\mathfrak{R}_3=0, \qquad -n_2\mathfrak{R}_1\times\mathfrak{R}_2+n_3\mathfrak{R}_1\times\mathfrak{R}_3=0, \qquad (4)$$

whence

$$\frac{\mathfrak{R}_2\times\mathfrak{R}_3}{n_1}=\frac{\mathfrak{R}_1\times\mathfrak{R}_3}{n_2}=\frac{\mathfrak{R}_1\times\mathfrak{R}_2}{n_3}. \qquad (5)$$

Our equation may therefore be regarded as signifying that the three vectors $\mathfrak{R}_1$, $\mathfrak{R}_2$, $\mathfrak{R}_3$ lie in one plane, and that the three triangles determined each by a pair of these vectors, and usually denoted by $[r_2r_3]$, $[r_1r_3]$, $[r_1r_2]$, are proportional to

$$A_1\left(1+\frac{B_1}{r_1^3}\right), \qquad \left(1-\frac{B_2}{r_2^3}\right), \qquad A_3\left(1+\frac{B_3}{r_3^3}\right).$$

Since this vector equation is equivalent to three ordinary equations, it is evidently sufficient to determine the three positions of the body in connection with the conditions that these positions must lie upon the lines of sight of three observations. To give analytical expression to these conditions, we may write $\mathfrak{E}_1$, $\mathfrak{E}_2$, $\mathfrak{E}_3$ for the vectors drawn from the sun to the three positions of the earth (or, more exactly, of the observatories where the observations have been made), $\mathfrak{F}_1$, $\mathfrak{F}_2$, $\mathfrak{F}_3$ for unit vectors drawn in the directions of the body, as observed, and ρ_1, ρ_2, ρ_3 for the three distances of the body from the places of observation. We have then

$$\mathfrak{R}_1=\mathfrak{E}_1+\rho_1\mathfrak{F}_1, \qquad \mathfrak{R}_2=\mathfrak{E}_2+\rho_2\mathfrak{F}_2, \qquad \mathfrak{R}_3=\mathfrak{E}_3+\rho_3\mathfrak{F}_3. \qquad (6)$$

By substitution of these values our fundamental equation becomes

$$A_1\left(1+\frac{B_1}{r_1^3}\right)(\mathfrak{E}_1+\rho_1\mathfrak{F}_1)-\left(1-\frac{B_2}{r_2^3}\right)(\mathfrak{E}_2+\rho_2\mathfrak{F}_2)+A_3\left(1+\frac{B_3}{r_3^3}\right)(\mathfrak{E}_3+\rho_3\mathfrak{F}_3)=0, \qquad (7)$$

where $\rho_1, \rho_2, \rho_3, r_1, r_2, r_3$ (the geocentric and heliocentric distances) are the only unknown quantities. From equations (6) we also get, by squaring both members in each,

$$r_1^2=\mathfrak{E}_1^2+2(\mathfrak{E}_1.\mathfrak{F}_1)\rho_1+\rho_1^2, \qquad r_2^2=\mathfrak{E}_2^2+2(\mathfrak{E}_2.\mathfrak{F}_2)\rho_2+\rho_2^2, \qquad r_3^2=\mathfrak{E}_3^2+2(\mathfrak{E}_3.\mathfrak{F}_3)\rho_3+\rho_3^2, \qquad (8)$$

by which the values of r_1, r_2, r_3 may be derived from those of ρ_1, ρ_2, ρ_3, or *vice versâ*. Equations (7) and (8), which are equivalent to six ordinary equations, are sufficient to determine the six quantities $r_1, r_2, r_3, \rho_1, \rho_2, \rho_3$; or, if we suppose the values of r_1, r_2, r_3 in terms of ρ_1, ρ_2, ρ_3 to be substituted in equation (7), we have a single vector equation, from which we may determine the three geocentric distances ρ_1, ρ_2, ρ_3.

It remains to be shown, first, how the numerical solution of the equation may be performed, and, secondly, how such an approximate solution of the actual problem may furnish the basis of a closer approximation.

SOLUTION OF THE FUNDAMENTAL EQUATION.

The relations with which we have to do will be rendered a little more simple if instead of each geocentric distance we introduce the distance of the body from the foot of the perpendicular from the sun upon the line of sight. If we set

$$q_1=\rho_1+(\mathfrak{E}_1\cdot\mathfrak{F}_1),\qquad q_2=\rho_2+(\mathfrak{E}_2\cdot\mathfrak{F}_2),\qquad q_3=\rho_3+(\mathfrak{E}_3\cdot\mathfrak{F}_3),\tag{9}$$

$$p_1^2=\mathfrak{E}_1^2-(\mathfrak{E}_1\cdot\mathfrak{F}_1)^2,\qquad p_2^2=\mathfrak{E}_2^2-(\mathfrak{E}_2\cdot\mathfrak{F}_2)^2,\qquad p_3=\mathfrak{E}_3^2-(\mathfrak{E}_3\cdot\mathfrak{F}_3)^2,\tag{10}$$

equations (8) become

$$r_1^2=q_1^2+p_1^2,\qquad r_2^2=q_2^2+p_2^2,\qquad r_3^2=q_3^2+p_3^2.\tag{11}$$

Let us also set, for brevity,

$$\mathfrak{S}_1=A_1\left(1+\frac{B_1}{r_1^3}\right)(\mathfrak{E}_1+\rho_1\mathfrak{F}_1),\qquad \mathfrak{S}_2=-\left(1-\frac{B_2}{r_2^3}\right)(\mathfrak{E}_2+\rho_2\mathfrak{F}_2),\qquad \mathfrak{S}_3=A_3\left(1+\frac{B_3}{r_3^3}\right)(\mathfrak{E}_3+\rho_3\mathfrak{F}_3).\tag{12}$$

Then $\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3$ may be regarded as functions respectively of ρ_1, ρ_2, ρ_3, therefore of q_1, q_3, q_3, and if we set

$$\mathfrak{S}'=\frac{d\mathfrak{S}_1}{dq_1},\qquad \mathfrak{S}''=\frac{d\mathfrak{S}_2}{dq_2},\qquad \mathfrak{S}'''=\frac{d\mathfrak{S}_3}{dq_3},\tag{13}$$

and

$$\mathfrak{S}=\mathfrak{S}_1+\mathfrak{S}_2+\mathfrak{S}_3,\tag{14}$$

we shall have

$$d\mathfrak{S}=\mathfrak{S}'dq_1+\mathfrak{S}''dq_2+\mathfrak{S}'''dq_3.\tag{15}$$

To determine the value of $\mathfrak{S}'$, we get by differentiation

$$\mathfrak{S}'=A_1\left(1+\frac{B_1}{r_1^3}\right)\mathfrak{F}_1-A_1\frac{3B_1}{r_1^4}\frac{dr_1}{dq_1}(\mathfrak{E}_1+\rho_1\mathfrak{F}_1).\tag{16}$$

But by (11)

$$\frac{dr_1}{dq_1}=\frac{q_1}{r_1}.\tag{17}$$

Therefore

$$\left.\begin{aligned}\mathfrak{S}' &=A_1\left(1+\frac{B_1}{r_1^3}\right)\mathfrak{F}_1-\frac{3B_1q_1}{r_1^5(1+B_1r_1^{-3})}\mathfrak{S}_1\\ \mathfrak{S}'' &=-\left(1-\frac{B_2}{r_2^3}\right)\mathfrak{F}_2+\frac{3B_2q_2}{r_2^5(1-B_2r_2^{-3})}\mathfrak{S}_2\\ \mathfrak{S}''' &=A_3\left(1+\frac{B_3}{r_3^3}\right)\mathfrak{F}_3-\frac{3B_3q_3}{r_3^5(1+B_3r_3^{-3})}\mathfrak{S}_3\end{aligned}\right\}\tag{18}$$

Now if any values of q_1, q_2, q_3 (either assumed or obtained by a previous approximation) give a certain residual $\mathfrak{S}$ (which would be zero if the values of q_1, q_2, q_3 satisfied the fundamental equation), and we wish to find the corrections $\Delta q_1, \Delta q_2, \Delta q_3$, which must be added to q_1, q_2, q_3

to reduce the residual to zero, we may apply equation (15) to these finite differences, and will have approximately, when these differences are not very large,

$$-\mathfrak{S}=\mathfrak{S}'\Delta q_1+\mathfrak{S}''\Delta q_2+\mathfrak{S}'''\Delta q_3. \qquad (19)$$

This gives*

$$\Delta q_1=-\frac{(\mathfrak{S}\mathfrak{S}''\mathfrak{S}''')}{(\mathfrak{S}'\mathfrak{S}''\mathfrak{S}''')} \qquad \Delta q_2=-\frac{(\mathfrak{S}\mathfrak{S}'''\mathfrak{S}')}{(\mathfrak{S}'\mathfrak{S}''\mathfrak{S}''')} \qquad \Delta q_3=-\frac{(\mathfrak{S}\mathfrak{S}'\mathfrak{S}'')}{(\mathfrak{S}'\mathfrak{S}''\mathfrak{S}''')}. \qquad (20)$$

From the corrected values of q_1, q_2, q_3 we may calculate a new residual $\mathfrak{S}$, and from that determine another correction for each of the quantities q_1, q_2, q_3.

It will sometimes be worth while to use formulæ a little less simple for the sake of a more rapid approximation. Instead of equation (19) we may write, with a higher degree of accuracy,

$$-\mathfrak{S}=\mathfrak{S}'\Delta q_1+\mathfrak{S}''\Delta q_2+\mathfrak{S}'''\Delta q_3+\tfrac{1}{2}\mathfrak{T}'(\Delta q_1)^2+\tfrac{1}{2}\mathfrak{T}''(\Delta q_2)^2+\tfrac{1}{2}\mathfrak{T}'''(\Delta q_3)^2, \qquad (21)$$

where

$$\left.\begin{aligned}
\mathfrak{T}' &=\frac{d^2\mathfrak{S}_1}{dq_1^2}=2A_1B_1\frac{d(r_1^{-3})}{dq_1}\mathfrak{z}_1+\frac{B_1}{1+B_1r_1^{-3}}\frac{d^2(r_1^{-3})}{dq_1^2}\mathfrak{S}_1\\
\mathfrak{T}'' &=\frac{d^2\mathfrak{S}_2}{dq_2^2}=\quad 2B_2\frac{d(r_2^{-3})}{dq_2}\mathfrak{z}_2-\frac{B_2}{1-B_2r_2^{-3}}\frac{d^2(r_2^{-3})}{dq_2^2}\mathfrak{S}_2\\
\mathfrak{T}''' &=\frac{d^2\mathfrak{S}_3}{dq_3^2}=2A_3B_3\frac{d(r_3^{-3})}{dq_3}\mathfrak{z}_3+\frac{B_3}{1+B_3r_3^{-3}}\frac{d^2(r_3^{-3})}{dq_3^2}\mathfrak{S}_3
\end{aligned}\right\} \qquad (22)$$

It is evident that $\mathfrak{T}''$ is generally many times greater than $\mathfrak{T}'$ or $\mathfrak{T}'''$, the factor B_2, in the case of equal intervals, being exactly ten times as great as A_1B_1 or A_3B_3. This shows, in the first place, that the accurate determination of Δq_2 is of the most importance for the subsequent approximations. It also shows that we may attain nearly the same accuracy in writing

$$-\mathfrak{S}=\mathfrak{S}'\Delta q_1+\mathfrak{S}''\Delta q_2+\mathfrak{S}'''\Delta q_3+\tfrac{1}{2}\mathfrak{T}''\Delta q_2^2 \qquad (23)$$

We may, however, often do a little better than this without using a more complicated equation. For $\mathfrak{T}'+\mathfrak{T}'''$ may be estimated very roughly as equal to $\frac{1}{5}\mathfrak{T}''$. Whenever, therefore, Δq_1 and Δq_3 are about as large as Δq_2, as is often the case, it may be a little better to use the coefficient $\frac{6}{10}$ instead of $\frac{1}{2}$ in the last term.

For Δq_2, then, we have the equation

$$-(\mathfrak{S}\mathfrak{S}'''\mathfrak{S}')=(\mathfrak{S}'\mathfrak{S}''\mathfrak{S}''')\Delta q_2+\tfrac{6}{10}(\mathfrak{T}''\mathfrak{S}'''\mathfrak{S}')\Delta q_2^2. \qquad (24)$$

$(\mathfrak{T}''\mathfrak{S}'''\mathfrak{S}')$ is easily computed from the formula

$$(\mathfrak{T}''\mathfrak{S}'''\mathfrak{S}')=\frac{1}{q_2}\left(1-5\frac{q_2^2}{r_2^2}\right)\Big((\mathfrak{S}'\mathfrak{S}''\mathfrak{S}''')+(\mathfrak{z}_2\mathfrak{S}'''\mathfrak{S}')\Big)-\frac{B_2}{q_2r_2^3}\left(1+\frac{q_2^2}{r_2^2}\right)(\mathfrak{z}_2\mathfrak{S}'''\mathfrak{S}'), \qquad (25)$$

which may be derived from equations (18) and (22).

The quadratic equation (24) gives two values of the correction to be applied to the position of the body. When they are not too large, they will belong to two different solutions of the problem, generally to the two least removed from the values assumed. But a very large value of Δq_2 must not be regarded as affording any trustworthy indication of a solution of the problem. In the majority of cases, we only care for one of the roots of the equation, which is distinguished by being very small, and which will be most easily calculated by a small correction to the value which we get by neglecting the quadratic term.†

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* These equations are obtained by taking the direct products of both members of the preceding equation with $\mathfrak{S}''\times\mathfrak{S}'''$, $\mathfrak{S}'''\times\mathfrak{S}'$, and $\mathfrak{S}'\times\mathfrak{S}''$, respectively. See foot-note on page 81.

† In the case of Swift's comet (V, 1880), the writer found by the quadratic equation −.247 and −.116 for corrections of the assumed geocentric distance .250. The first of these numbers gives an approximation to the position of the earth; the second to that of the comet, viz., the geocentric distance .134 instead of the true value .1333. The coefficient $\frac{6}{10}$ was used in the quadratic equation; with the coefficient $\frac{1}{2}$ the approximations would not be quite so good. The value of the correction obtained by neglecting the quadratic term was .079, which indicates that the approximations (in this very critical case) would be quite tedious without the use of the quadratic term.

When a comet is somewhat near the earth we may make use of the fact that the earth's orbit is one solution of the problem, *i. e.*, that $-\rho_2$ is one value of Δq_2, to save the trifling labor of computing the value of $(\mathfrak{T}''\mathfrak{S}'''\mathfrak{S}')$. For it is evident from the theory of equations that if $-\rho_2$ and z are the two roots,

$$\rho_2 - z = \frac{(\mathfrak{S}'\mathfrak{S}''\mathfrak{S}''')}{\frac{3}{5}(\mathfrak{T}''\mathfrak{S}'''\mathfrak{S}')} \qquad -\rho_2 z = \frac{(\mathfrak{S}\mathfrak{S}'''\mathfrak{S}')}{\frac{3}{5}(\mathfrak{T}''\mathfrak{S}'''\mathfrak{S}')}.$$

Eliminating $(\mathfrak{T}''\mathfrak{S}'''\mathfrak{S}')$, we have

$$(\rho_2 - z)(\mathfrak{S}\mathfrak{S}'''\mathfrak{S}') = -\rho_2 z(\mathfrak{S}'\mathfrak{S}''\mathfrak{S}'''),$$

whence

$$\frac{1}{z} = \frac{1}{\rho_2} - \frac{(\mathfrak{S}'\mathfrak{S}''\mathfrak{S}''')}{(\mathfrak{S}\mathfrak{S}'''\mathfrak{S}')}.$$

Now $-\frac{(\mathfrak{S}\mathfrak{S}'''\mathfrak{S}')}{(\mathfrak{S}'\mathfrak{S}''\mathfrak{S}''')}$ is the value of Δq_2, which we obtain if we neglect the quadratic term in equation (24). If we call this value $[\Delta q_2]$, we have for the more exact value*

$$\Delta q_2 = \frac{[\Delta q_2]}{1 + \frac{[\Delta q_2]}{\rho_2}} \tag{26}$$

The quantities Δq_1 and Δq_3 might be calculated by the equations

$$\left.\begin{aligned} -(\mathfrak{S}\mathfrak{S}''\mathfrak{S}''') &= (\mathfrak{S}'\mathfrak{S}''\mathfrak{S}''')\Delta q_1 + \tfrac{6}{10}(\mathfrak{T}''\mathfrak{S}''\mathfrak{S}''')\Delta q_2^2 \\ -(\mathfrak{S}\mathfrak{S}'\mathfrak{S}'') &= (\mathfrak{S}'\mathfrak{S}''\mathfrak{S}''')\Delta q_3 + \tfrac{6}{10}(\mathfrak{T}''\mathfrak{S}'\mathfrak{S}'')\,\Delta q_2^2 \end{aligned}\right\} \tag{27}$$

But a little examination will show that the coefficients of Δq_2^2 in these equations will not generally have very different values from the coefficient of the same quantity in equation (24). We may therefore write with sufficient accuracy

$$\Delta q_1 = [\Delta q_1] + \Delta q_2 - [\Delta q_2], \qquad \Delta q_3 = [\Delta q_3] + \Delta q_2 - [\Delta q_2], \tag{28}$$

where $[\Delta q_1]$, $[\Delta q_2]$, $[\Delta q_3]$ denote the values obtained from equations (20).

In making successive corrections of the distances q_1, q_2, q_3 it will not be necessary to recalculate the values of $\mathfrak{S}'$, $\mathfrak{S}''$, $\mathfrak{S}'''$, when these have been calculated from fairly good values of q_1, q_2, q_3. But when, as is generally the case, the first assumption is only a rude guess, the values of $\mathfrak{S}'$, $\mathfrak{S}''$, $\mathfrak{S}'''$ should be recalculated after one or two corrections of q_1, q_2, q_3. To get the best results when we do not recalculate $\mathfrak{S}'$, $\mathfrak{S}''$, $\mathfrak{S}'''$, we may proceed as follows: Let $\mathfrak{S}'$, $\mathfrak{S}''$, $\mathfrak{S}'''$ denote the values which have been calculated; Dq_1, Dq_2, Dq_3, respectively, the sum of the corrections of each of the quantities q_1, q_2, q_3, which have been made since the calculation of $\mathfrak{S}'$, $\mathfrak{S}''$, $\mathfrak{S}'''$; $\mathfrak{S}$ the residual after all the corrections of q_1, q_2, q_3, which have been made; and Δq_1, Δq_2, Δq_3 the remaining corrections which we are seeking. We have, then, very nearly

$$-\mathfrak{S} = \{\mathfrak{S}' + \mathfrak{T}'(Dq_1 + \tfrac{1}{2}\Delta q_1)\}\Delta q_1 + \{\mathfrak{S}'' + \mathfrak{T}''(Dq_2 + \tfrac{1}{2}\Delta q_2)\}\Delta q_2 + \{\mathfrak{S}''' + \mathfrak{T}'''(Dq_3 + \tfrac{1}{2}\Delta q_3)\}\Delta q_3. \tag{29}$$

The same considerations which we applied to equation (21) enable us to simplify this equation also, and to write with a fair degree of accuracy

$$-(\mathfrak{S}\mathfrak{S}'''\mathfrak{S}') = \{(\mathfrak{S}'\mathfrak{S}''\mathfrak{S}''') + \tfrac{3}{5}(\mathfrak{T}''\mathfrak{S}'''\mathfrak{S}')(Dq_2 + \tfrac{1}{2}\Delta q_2)\}\Delta q_2, \tag{30}$$

$$\Delta q_1 = [\Delta q_1] + \Delta q_2 - [\Delta q_2], \qquad \Delta q_3 = [\Delta q_3] + \Delta q_2 - [\Delta q_2], \tag{31}$$

where

$$[\Delta q_1] = -\frac{(\mathfrak{S}\mathfrak{S}''\mathfrak{S}''')}{(\mathfrak{S}'\mathfrak{S}''\mathfrak{S}''')} \qquad [\Delta q_2] = -\frac{(\mathfrak{S}\mathfrak{S}'''\mathfrak{S}')}{(\mathfrak{S}'\mathfrak{S}''\mathfrak{S}''')} \qquad [\Delta q_3] = -\frac{(\mathfrak{S}\mathfrak{S}'\mathfrak{S}'')}{(\mathfrak{S}'\mathfrak{S}''\mathfrak{S}''')}. \tag{32}$$

*In the case mentioned in the preceding foot-note, from $[\Delta q_2] = -.079$ and $\rho_2 = .25$, we get $\Delta q_2 = -.1155$, which is sensibly the same value as that obtained by calculating the quadratic term.

CORRECTION OF THE FUNDAMENTAL EQUATION.

When we have thus determined, by the numerical solution of our fundamental equation, approximate values of the three positions of the body, it will always be possible to apply a small numerical correction to the equation, so as to make it agree exactly with the laws of elliptic motion in a fictitious case differing but little from the actual. After such a correction, the equation will evidently apply to the actual case with a much higher degree of approximation.

There is room for great diversity in the application of this principle. The method which appears to the writer the most simple and direct is the following, in which the correction of the intervals for aberration is combined with the correction required by the approximate nature of the equation.*

The solution of the fundamental equation gives us three points, which must necessarily lie in one plane with the sun, and in the lines of sight of the several observations. Through these points we may pass an ellipse, and calculate the intervals of time required by the exact laws of elliptic motion for the passage of the body between them. If these calculated intervals should be identical with the given intervals, corrected for aberration, we would evidently have the true solution of the problem. But suppose, to fix our ideas, that the calculated intervals are a little too long. It is evident that if we repeat our calculations, using in our fundamental equation intervals shortened in the same ratio as the calculated intervals have come out too long, the intervals calculated from the second solution of the fundamental equation must agree almost exactly with the desired values. If necessary, this process may be repeated, and thus any required degree of accuracy may be obtained, whenever the solution of the uncorrected equation gives an approximation to the true positions. For this it is necessary that the intervals should not be too great. It appears, however, from the results of the example of Ceres, given hereafter, in which the heliocentric motion exceeds 62°, but the calculated values of the intervals of time differ from the given values by little more than one part in two thousand, that we have here not approached the limit of the application of our formula.

In the usual terminology of the subject, the fundamental equation with intervals uncorrected for aberration represents the *first hypothesis*, the same equation with the intervals affected by certain numerical coefficients (differing little from unity) represents the *second hypothesis*, the *third hypothesis*, should such be necessary, is represented by a similar equation, with corrected coefficients, etc.

In the process indicated there are certain economies of labor which should not be left unmentioned, and certain precautions to be observed in order that the neglected figures in our computations may not unduly influence the result.

It is evident, in the first place, that for the correction of our fundamental equation we need not trouble ourselves with the position of the orbit in the solar system. The intervals of time, which determine this correction, depend only on the three heliocentric distances r_1, r_2, r_3 and the two heliocentric angles, which will be represented by v_2-v_1 and v_3-v_2, if we write v_1, v_2, v_3 for the true anomalies. These angles (v_2-v_1 and v_3-v_2) may be determined from r_1, r_2, r_3 and n_1, n_2, n_3, and therefore from r_1, r_2, r_3 and the given intervals. For our fundamental equation, which may be written

$$n_1\mathfrak{R}_1-n_2\mathfrak{R}_2+n_3\mathfrak{R}_3=0, \tag{33}$$

indicates that we may form a triangle in which the lengths of the sides shall be n_1r_1, n_2r_2, and n_3r_3, (let us say for brevity, s_1, s_2, s_3,) and the directions of the sides parallel with the three heliocentric directions of the body. The angles opposite s_1 and s_3 will be respectively v_3-v_2 and v_2-v_1. We have therefore, by a well-known formula,

$$\left.\begin{aligned}\tan\frac{v_3-v_2}{2}&=\sqrt{\frac{(s_1-s_2+s_3)(s_1+s_2-s_3)}{(s_1+s_2+s_3)(-s_1+s_2+s_3)}}\\ \tan\frac{v_2-v_1}{2}&=\sqrt{\frac{(-s_1+s_2+s_3)(s_1-s_2+s_3)}{(s_1+s_2+s_3)(s_1+s_2-s_3)}}\end{aligned}\right\} \tag{34}$$

* When an approximate orbit is known in advance, we may correct the fundamental equation at once. The formulæ will be given in the Summary, § XII.

As soon, therefore, as the solution of our fundamental equation has given a sufficient approximation to the values of r_1, r_2, r_3 (say five- or six-figure values, if our final result is to be as exact as seven-figure logarithms can make it), we calculate n_1, n_2, n_3 with seven-figure logarithms by equations (2), and the heliocentric angles by equations (34).

The semi-parameter corresponding to these values of the heliocentric distances and angles is given by the equation

$$p=\frac{n_1 r_1-n_2 r_2+n_3 r_3}{n_1-n_2+n_3} \tag{35}$$

The expression $n_1-n_2+n_3$, which occurs in the value of the semi-parameter, and the expression $n_1 r_1-n_2 r_2+n_3 r_3$, or $s_1-s_2+s_3$, which occurs both in the value of the semi-parameter and in the formulæ for determining the heliocentric angles, represent small quantities of the second order (if we call the heliocentric angles small quantities of the first order), and cannot be very accurately determined from approximate numerical values of their separate terms. The first of these quantities may, however, be determined accurately by the formula

$$n_1-n_2+n_3=\frac{A_1 B_1}{r_1^3}+\frac{B_2}{r_2^3}+\frac{A_3 B_3}{r_3^3} \tag{36}$$

With respect to the quantity $s_1-s_2+s_3$, a little consideration will show that if we are careful to use the same value wherever the expression occurs, both in the formulæ for the heliocentric angles and for the semi-parameter, the inaccuracy of the determination of this value from the cause mentioned will be of no consequence in the process of correcting the fundamental equation. For, although the logarithm of $s_1-s_2+s_3$ as calculated by seven figure logarithms from r_1, r_2, r_3 may be accurate only to four or five figures, we may regard it as absolutely correct if we make a very small change in the value of one of the heliocentric distances (say r_2). We need not trouble ourselves farther about this change, for it will be of a magnitude which we neglect in computations with seven-figure tables. That the heliocentric angles thus determined may not agree as closely as they might with the positions on the lines of sight determined by the first solution of the fundamental equation is of no especial consequence in the correction of the fundamental equation, which only requires the exact fulfillment of two conditions, viz., that our values of the heliocentric distances and angles shall have the relations required by the fundamental equation to the given intervals of time, and that they shall have the relations required by the exact laws of elliptic motion to the calculated intervals of time. The third condition, that none of these values shall differ too widely from the actual values, is of a looser character.

After the determination of the heliocentric angles and the semi-parameter, the eccentricity and the true anomalies of the three positions may next be determined, and from these the intervals of time. These processes require no especial notice. The appropriate formulæ will be given in the Summary of Formulæ.

DETERMINATION OF THE ORBIT FROM THE THREE POSITIONS AND THE INTERVALS OF TIME.

The values of the semi-parameter and the heliocentric angles as given in the preceding paragraphs depend upon the quantity $s_1-s_2+s_3$, the numerical determination of which from s_1, s_2, and s_3 is critical to the second degree when the heliocentric angles are small. This was of no consequence in the process which we have called the *correction of the fundamental equation*. But for the actual determination of the orbit from the positions given by the corrected equation—or by the uncorrected equation, when we judge that to be sufficient—a more accurate determination of this quantity will generally be necessary. This may be obtained in different ways, of which the following is perhaps the most simple. Let us set

$$\mathfrak{S}_4=\mathfrak{S}_3-\mathfrak{S}_1, \tag{37}$$

and s_4 for the length of the vector $\mathfrak{S}_4$, obtained by taking the square root of the sum of the squares of the components of the vector. It is evident that s_2 is the longer and s_4 the shorter diagonal of a parallelogram of which the sides are s_1 and s_3. The area of the triangle having the sides s_1, s_2, s_3 is therefore equal to that of the triangle having the sides s_1, s_3, s_4, each being one-half of the parallelogram. This gives

$$(s_1+s_2+s_3)(-s_1+s_2+s_3)(s_1-s_2+s_3)(s_1+s_2-s_3)=(s_1+s_4+s_3)(-s_1+s_4+s_3)(s_1-s_4+s_3)(s_1+s_4-s_3), \quad (38)$$

and

$$s_1-s_2+s_3=\frac{(s_1+s_4+s_3)(-s_1+s_4+s_3)(s_1-s_4+s_3)(s_1+s_4-s_3)}{(s_1+s_2+s_3)(-s_1+s_2+s_3)(s_1+s_2-s_3)}. \quad (39)$$

The numerical determination of this value of $s_1-s_2+s_3$ is critical only to the first degree.

The eccentricity and the true anomalies may be determined in the same way as in the correction of the formula. The position of the orbit in space may be derived from the following considerations. The vector $-\mathfrak{S}_2$ is directed from the sun toward the second position of the body; the vector $\mathfrak{S}_4$ from the first to the third position. If we set

$$\mathfrak{S}_5=\mathfrak{S}_4-\frac{\mathfrak{S}_4\cdot\mathfrak{S}_2}{s_2^2}\mathfrak{S}_2, \quad (40)$$

the vector $\mathfrak{S}_5$ will be in the plane of the orbit, perpendicular to $-\mathfrak{S}_2$ and on the side toward which anomalies increase. If we write s_5 for the length of $\mathfrak{S}_5$,

$$-\frac{\mathfrak{S}_2}{s_2} \text{ and } \frac{\mathfrak{S}_5}{s_5}$$

will be unit vectors. Let $\mathfrak{Z}$ and $\mathfrak{Z}'$ be unit vectors determining the position of the orbit, $\mathfrak{Z}$ being drawn from the sun toward the perihelion, and $\mathfrak{Z}'$ at right angles to $\mathfrak{Z}$, in the plane of the orbit, and on the side toward which anomalies increase. Then

$$\mathfrak{Z}=-\cos v_2\frac{\mathfrak{S}_2}{s_2}-\sin v_2\frac{\mathfrak{S}_5}{s_5} \quad (41)$$

$$\mathfrak{Z}'=-\sin v_2\frac{\mathfrak{S}_2}{s_2}+\cos v_2\frac{\mathfrak{S}_5}{s_5} \quad (42)$$

The time of perihelion passage (T) may be determined from any one of the observations by the equation

$$\frac{k}{a^{\frac{3}{2}}}(t-T)=E-e\sin E, \quad (43)$$

the eccentric anomaly E being calculated from the true anomaly v. The interval $t-T$ in this equation is to be measured in days. A better value of T may be found by averaging the three values given by the separate observations, with such weights as the circumstances may suggest. But any considerable differences in the three values of T would indicate the necessity of a second correction of the formula, and furnish the basis for it.

For the calculation of an ephemeris we have

$$\mathfrak{R}=-ae\mathfrak{Z}+\cos E\, a\mathfrak{Z}+\sin E\, b\mathfrak{Z}' \quad (44)$$

in connection with the preceding equation.

Sometimes it may be worth while to make the calculations for the correction of the formula in the slightly longer form indicated for the determination of the orbit. This will be the case when we wish simultaneously to correct the formula for its theoretical imperfection, and to correct the observations by comparison with others not too remote. The rough approximation to the orbit given by the uncorrected formula may be sufficient for this purpose. In fact, for observations separated by very small intervals, the imperfection of the uncorrected formula will be likely to affect the orbit less than the errors of the observations.

The computer may prefer to determine the orbit from the first and third heliocentric positions with their times. This process, which has certain advantages, is perhaps a little longer than that here given, and does not lend itself quite so readily to successive improvements of the hypothesis. When it is desired to derive an improved hypothesis from an orbit thus determined, the formulæ in § XII of the summary may be used.

SUMMARY OF FORMULÆ

WITH DIRECTIONS FOR USE.

[For the case in which an approximate orbit is known in advance, see XII.]

I.

Preliminary computations relating to the intervals of time.

t_1, t_2, t_3=times of the observations in days.

$\log k = 8.2355814$ (after Gauss)

$$\tau_1 = k(t_3 - t_2) \qquad \tau_3 = k(t_2 - t_1)$$

$$A_1 = \frac{t_3 - t_2}{t_3 - t_1} \qquad A_3 = \frac{t_2 - t_1}{t_3 - t_1}$$

$$B_1 = \frac{-\tau_1^2 + \tau_1\tau_3 + \tau_3^2}{12} \qquad B_2 = \frac{\tau_1^2 + 3\tau_1\tau_3 + \tau_3^2}{12} \qquad B_3 = \frac{\tau_1^2 + \tau_1\tau_3 - \tau_3^2}{12}$$

For control: $$A_1 B_1 + B_2 + A_3 B_3 = \tfrac{1}{2}\tau_1\tau_3$$

II.

Preliminary computations relating to the first observation.

X_1, Y_1, Z_1 (components of $\mathfrak{E}_1$)=the heliocentric coördinates of the earth, increased by the geocentric coördinates of the observatory.

ξ_1, η_1, ζ_1 (components of $\mathfrak{Z}_1$)=the direction-cosines of the observed position, corrected for the aberration of the fixed stars.

$$\mathfrak{E}_1^2 = X_1^2 + Y_1^2 + Z_1^2 \qquad (\mathfrak{E}_1 \cdot \mathfrak{Z}_1) = X_1\xi_1 + Y_1\eta_1 + Z_1\zeta_1 \qquad p_1^2 = \mathfrak{E}_1^2 - (\mathfrak{E}_1 \cdot \mathfrak{Z}_1)^2$$

Preliminary computations relating to the second and third observations.

The formulæ are entirely analogous to those relating to the first observation, the quantities being distinguished by the proper suffixes.

III.

Equations of the first hypothesis.

When the preceding quantities have been computed, their numerical values (or their logarithms, when more convenient for computation,) are to be substituted in the following equations:

Components of $\mathfrak{S}_1$

$$q_1 = \rho_1 + (\mathfrak{E}_1 \cdot \mathfrak{Z}_1) \qquad \alpha_1 = A_1\xi_1(1 + R_1)\left(q_1 + \frac{X_1}{\xi_1} - (\mathfrak{E}_1 \cdot \mathfrak{Z}_1)\right)$$

$$r_1^2 = q_1^2 + p_1^2 \qquad \beta_1 = A_1\eta_1(1 + R_1)\left(q_1 + \frac{Y_1}{\eta_1} - (\mathfrak{E}_1 \cdot \mathfrak{Z}_1)\right) \quad \text{III}_1$$

$$R_1 = \frac{B_1}{r_1^3} \qquad \gamma_1 = A_1\zeta_1(1 + R_1)\left(q_1 + \frac{Z_1}{\zeta_1} - (\mathfrak{E}_1 \cdot \mathfrak{Z}_1)\right)$$

For control: $$s_1^2 = \alpha_1^2 + \beta_1^2 + \gamma_1^2 = A_1^2(1 + R_1)^2 r_1^2$$

Components of $\mathfrak{S}'$

$$P'=\frac{3R_1q_1}{(1+R_1)r_1^2} \qquad \left.\begin{aligned} \alpha'&=A_1\xi_1+A_1\xi_1R_1-P'\alpha_1\\ \beta'&=A_1\eta_1+A_1\eta_1R_1-P'\beta_1\\ \gamma'&=A_1\zeta_1+A_1\zeta_1R_1-P'\gamma_1\end{aligned}\right\}\text{III}'$$

Components of $\mathfrak{S}_2$

$$q_2=\rho_2+(\mathfrak{E}_2\cdot\mathfrak{F}_2) \qquad \alpha_2=-\xi_2(1-R_2)\left(q_2+\frac{X_2}{\xi_2}-(\mathfrak{E}_2\cdot\mathfrak{F}_2)\right)$$

$$r_2^2=q_2^2+p_2^2 \qquad \beta_2=-\eta_2(1-R_2)\left(q_2+\frac{Y_2}{\eta_2}-(\mathfrak{E}_2\cdot\mathfrak{F}_2)\right) \quad \text{III}_2$$

$$R_2=\frac{B_2}{r_2^3} \qquad \gamma_2=-\zeta_2(1-R_2)\left(q_2+\frac{Z_2}{\zeta_2}-(\mathfrak{E}_2\cdot\mathfrak{F}_2)\right)$$

For control:

$$s_2^2=\alpha_2^2+\beta_2^2+\gamma_2^2=(1-R_2)^2r_2^2$$

Components of $\mathfrak{S}''$

$$P''=\frac{3R_2q_2}{(1-R_2)r_2^2} \qquad \left.\begin{aligned} \alpha''&=-\xi_2+\xi_2R_2+P''\alpha_2\\ \beta''&=-\eta_2+\eta_2R_2+P''\beta_2\\ \gamma''&=-\zeta_2+\zeta_2R_2+P''\gamma_2\end{aligned}\right\}\text{III}''$$

Components of $\mathfrak{S}_3$

$$q_3=\rho_3+(\mathfrak{E}_3\cdot\mathfrak{F}_3) \qquad \alpha_3=A_3\xi_3(1+R_3)\left(q_3+\frac{X_3}{\xi_3}-(\mathfrak{E}_3\cdot\mathfrak{F}_3)\right)$$

$$r_3^2=q_3^2+p_3^2 \qquad \beta_3=A_3\eta_3(1+R_3)\left(q_3+\frac{Y_3}{\eta_3}-(\mathfrak{E}_3\cdot\mathfrak{F}_3)\right) \quad \text{III}_3$$

$$R_3=\frac{B_3}{r_3^3} \qquad \gamma_3=A_3\zeta_3(1+R_3)\left(q_3+\frac{Z_3}{\zeta_3}-(\mathfrak{E}_3\cdot\mathfrak{F}_3)\right)$$

For control:

$$s_3^2=\alpha_3^2+\beta_3^2+\gamma_3^2=A_3^2(1+R_3)^2r_3^2$$

Components of $\mathfrak{S}'''$

$$P'''=\frac{3R_3q_3}{(1+R_3)r_3^2} \qquad \left.\begin{aligned} \alpha'''&=A_3\xi_3+A_3\xi_3R_3-P'''\alpha_3\\ \beta'''&=A_3\eta_3+A_3\eta_3R_3-P'''\beta_3\\ \gamma'''&=A_3\zeta_3+A_3\zeta_3R_3-P'''\gamma_3\end{aligned}\right\}\text{III}'''$$

The computer is now to assume any reasonable values either of the geocentric distances, ρ_1, ρ_2, ρ_3, or of the heliocentric distances, r_1, r_2, r_3, (the former in the case of a comet, the latter in the case of an asteroid,) and from these assumed values to compute the rest of the following quantities:

By equations III_1, III'.	By equations III_2, III''.	By equations III_3, III'''.
q_1	q_2	q_3
$\log r_1$	$\log r_2$	$\log r_3$
$\log R_1$	$\log R_2$	$\log R_3$
$\log (1+R_1)$	$\log (1-R_2)$	$\log (1+R_3)$
$\log P'$	$\log P''$	$\log P'''$
α_1	α_2	α_3
β_1	β_2	β_3
γ_1	γ_2	γ_3
α'	α''	α'''
β'	β''	β'''
γ'	γ''	γ'''

IV.

Calculations relating to differential coefficients.

Components of $\mathfrak{S}'' \times \mathfrak{S}'''$	Components of $\mathfrak{S}''' \times \mathfrak{S}'$	Components of $\mathfrak{S}' \times \mathfrak{S}''$
$a_1=\beta''\gamma'''-\gamma''\beta'''$	$a_2=\beta'''\gamma'-\gamma'''\beta'$	$a_3=\beta'\gamma''-\gamma'\beta''$
$b_1=\gamma''\alpha'''-\alpha''\gamma'''$	$b_2=\gamma'''\alpha'-\alpha'''\gamma'$	$b_3=\gamma'\alpha''-\alpha'\gamma''$
$c_1=\alpha''\beta'''-\beta''\alpha'''$	$c_2=\alpha'''\beta'-\beta'''\alpha'$	$c_3=\alpha'\beta''-\beta'\alpha''$

$$G=(\mathfrak{S}'\mathfrak{S}''\mathfrak{S}''')=a_1\alpha'+b_1\beta'+c_1\gamma'=a_2\alpha''+b_2\beta''+c_2\gamma''=a_3\alpha'''+b_3\beta'''+c_3\gamma'''$$

These computations are controlled by the agreement of the three values of G.

The following are not necessary except when the corrections to be made are large:

$$H=(\mathfrak{Z}_2\mathfrak{S}'''\mathfrak{S}')=a_2\xi_2+b_2\eta_2+c_2\zeta_2$$

$$L=\frac{1}{q_2}\left(1+\frac{H}{G}\right)\left(1-5\frac{q_2^2}{r_2^2}\right)-\frac{R_2H}{q_2G}\left(1+\frac{q_2^2}{r_2^2}\right)$$

V.

Corrections of the geocentric distances.

Components of $\mathfrak{S}$.

$$\alpha=\alpha_1+\alpha_2+\alpha_3 \qquad C_1=-\frac{a_1\alpha+b_1\beta+c_1\gamma}{G}$$

$$\beta=\beta_1+\beta_2+\beta_3 \qquad C_2=-\frac{a_2\alpha+b_2\beta+c_2\gamma}{G}$$

$$\gamma=\gamma_1+\gamma_2+\gamma_3 \qquad C_3=-\frac{a_3\alpha+b_3\beta+c_3\gamma}{G}$$

$$\varDelta q_2=C_2-\tfrac{6}{10}L(\varDelta q_2)^2$$

(This equation will generally be most easily solved by repeated substitutions.)

$$\varDelta q_1=C_1-\tfrac{6}{10}L(\varDelta q_2)^2 \qquad \varDelta q_3=C_3-\tfrac{6}{10}L(\varDelta q_2)^2$$

VI.

Successive corrections.

$\varDelta q_1$, $\varDelta q_2$, $\varDelta q_3$ are to be added as corrections to q_1, q_2, q_3. With the new values thus obtained the computation by equations III_1, III_2, III_3 are to be recommenced. Two courses are now open:

(*a*) The work may be carried on exactly as before to the determination of new corrections for q_1, q_2, q_3.

(*b*) The computations by equations III′, III″, III‴, and IV may be omitted, and the old values of a_1, b_1, c_1, a_2, etc., G, and L may be used with the new residuals α, β, γ to get new corrections for q_1, q_2, q_3 by the equations

$$\varDelta q_2=\frac{C_2}{1+\frac{6}{5}L(Dq_2+\frac{1}{2}C_2)},$$

$$\varDelta q_1=C_1+\varDelta q_2-C_2, \qquad \varDelta q_3=C_3+\varDelta q_2-C_2,$$

where Dq_2 denotes the former correction of q_2. (More generally, at any stage of the work, Dq_2 will represent the sum of all the corrections of q_2 which have been made since the last computation of a_1, b_1, etc.)

So far as any general rule can be given, it is advised to recompute a_1, b_1, etc., and G *once*, perhaps after the second corrections of q_1, q_2, q_3, unless the assumed values represent a fair approximation. Whether L is also to be recomputed, depends on its magnitude and on that of the correction of q_2, which remains to be made. In the later stages of the work, when the corrections are small, the terms containing L may be neglected altogether.

The corrections of q_1, q_2, q_3 should be repeated until the equations

$$\alpha=0 \qquad \beta=0 \qquad \gamma=0$$

are nearly satisfied. Approximate values of r_1, r_2, r_3 may suffice for the following computations, which, however, must be made with the greatest exactness.

VII.

Test of the first hypothesis.

$\log r_1$, $\log r_2$, $\log r_3$, (approximate values from the preceding computations.)

$$N=A_1B_1r_1^{-3}+B_2r_2^{-3}+A_3B_3r_3^{-3}$$
$$s_1=A_1r_1+A_1B_1r_1^{-2}$$
$$s_2=r_2-B_2r_2^{-2}$$
$$s_3=A_3r_3+A_3B_3r_3^{-2}$$
$$s=\tfrac{1}{2}(s_1+s_2+s_3)$$
$$s-s_1,\ s-s_2,\ s-s_3$$

The value of $s-s_2$ may be very small, and its logarithm in consequence ill determined. This will do no harm if the computer is careful to use the same value—computed, of course, as carefully as possible—wherever the expression occurs in the following formulæ.

$$\mathrm{R}=\sqrt{\frac{(s-s_1)(s-s_2)(s-s_3)}{s}} \qquad \tan\tfrac{1}{2}(v_2-v_1)=\frac{\mathrm{R}}{s-s_3}$$

$$v=\frac{2(s-s_2)}{N} \qquad \tan\tfrac{1}{2}(v_3-v_2)=\frac{\mathrm{R}}{s-s_1}$$

$$\tan\tfrac{1}{2}(v_3-v_1)=\frac{s-s_2}{\mathrm{R}}$$

For adjustment of values: $\quad \tfrac{1}{2}(v_3-v_1)=\tfrac{1}{2}(v_2-v_1)+\tfrac{1}{2}(v_3-v_2)$

$$e\sin\tfrac{1}{2}(v_3+v_1)=\frac{\frac{p}{r_1}-\frac{p}{r_3}}{2\sin\tfrac{1}{2}(v_3-v_1)}$$

$$e\cos\tfrac{1}{2}(v_3+v_1)=\frac{\frac{p}{r_1}+\frac{p}{r_3}-2}{2\cos\tfrac{1}{2}(v_3-v_1)}$$

$$\tan\tfrac{1}{2}(v_3+v_1) \qquad e^2$$

For control: $\quad e\cos v_2=\frac{p}{r_2}-1$

$$\varepsilon=\sqrt{\frac{1-e}{1+e}} \qquad a=\frac{p}{1-e^2}$$

$$\tan\tfrac{1}{2}E_1=\varepsilon\tan\tfrac{1}{2}v_1 \qquad \tan\tfrac{1}{2}E_2=\varepsilon\tan\tfrac{1}{2}v_2 \qquad \tan\tfrac{1}{2}E_3=\varepsilon\tan\tfrac{1}{2}v_3$$

$$\tau_{1\,\mathrm{calc.}}=a^{\frac{3}{2}}(E_3-E_2)+ea^{\frac{3}{2}}\sin E_2-ea^{\frac{3}{2}}\sin E_3$$

$$\tau_{3\,\mathrm{calc.}}=a^{\frac{3}{2}}(E_2-E_1)+ea^{\frac{3}{2}}\sin E_1-ea^{\frac{3}{2}}\sin E_2$$

VIII.

For the second hypothesis.

$$\delta\tau_1 = .0057613k(\rho_2 - \rho_3)$$
$$\delta\tau_3 = .0057613k(\rho_1 - \rho_2)$$

(aberration-constant after Struve.)
$\log(.0057613k) = 5.99610$

$$\Delta \log \tau_1 = \log \tau_1 - \log(\tau_{1\,\text{calc.}} - \delta\tau_1)$$
$$\Delta \log \tau_3 = \log \tau_3 - \log(\tau_{3\,\text{calc.}} - \delta\tau_3)$$

$$\Delta \log(\tau_1\tau_3) = \Delta \log \tau_1 + \Delta \log \tau_3$$
$$\Delta \log \frac{\tau_1}{\tau_3} = \Delta \log \tau_1 - \Delta \log \tau_3$$

$$\Delta \log A_1 = -A_3 \Delta \log \frac{\tau_1}{\tau_3}$$
$$\Delta \log A_3 = -A_1 \Delta \log \frac{\tau_1}{\tau_3}$$

$$\Delta \log B_1 = \Delta \log(\tau_1\tau_3) - \frac{\tau_1^2 + \tau_3^2}{12B_1} \Delta \log \frac{\tau_1}{\tau_3}$$
$$\Delta \log B_2 = \Delta \log(\tau_1\tau_3) + \frac{\tau_1^2 - \tau_3^2}{12B_2} \Delta \log \frac{\tau_1}{\tau_3}$$
$$\Delta \log B_3 = \Delta \log(\tau_1\tau_3) + \frac{\tau_1^2 + \tau_3^2}{12B_3} \Delta \log \frac{\tau_1}{\tau_3}$$

These corrections are to be added to the logarithms of A_1, A_3, B_1, B_2, B_3, in equations III_1, III_2, III_3, and the corrected equations used to correct the values of q_1, q_2, q_3, until the residuals α, β, γ vanish. The new values of A_1, A_3 must satisfy the relation $A_1 + A_3 = 1$, and the corrections $\Delta \log A_1$, $\Delta \log A_3$ must be adjusted, if necessary, for this end.

Third hypothesis.

A second correction of equations III_1, III_2, III_3 may be obtained in the same manner as the first. but this will rarely be necessary.

IX.

Determination of the ellipse.

It is supposed that the values of

$$\alpha_1,\ \beta_1,\ \gamma_1, \qquad \alpha_2,\ \beta_2,\ \gamma_2, \qquad \alpha_3,\ \beta_3,\ \gamma_3,$$
$$r_1,\ r_2,\ r_3, \qquad R_1,\ R_2,\ R_3, \qquad s_1,\ s_2,\ s_3,$$

have been computed by equations III_1, III_2, III_3 with the greatest exactness, so as to make the residuals α, β, γ vanish, and that the two formulæ for each of the quantities s_1, s_2, s_3 give sensibly the same value.

Components of $\mathfrak{S}_4$

$$\alpha_4 = \alpha_3 - \alpha_1$$
$$\beta_4 = \beta_3 - \beta_1$$
$$\gamma_4 = \gamma_3 - \gamma_1$$
$$s_4^2 = \alpha_4^2 + \beta_4^2 + \gamma_4^2$$
$$s = \tfrac{1}{2}(s_1 + s_2 + s_3)$$

Components of $\mathfrak{S}_5$

$$\alpha_5 = \alpha_4 - \frac{\alpha_4\alpha_2 + \beta_4\beta_2 + \gamma_4\gamma_2}{s_2^2}\alpha_2$$
$$\beta_5 = \beta_4 - \frac{\alpha_4\alpha_2 + \beta_4\beta_2 + \gamma_4\gamma_2}{s_2^2}\beta_2$$
$$\gamma_5 = \gamma_4 - \frac{\alpha_4\alpha_2 + \beta_4\beta_2 + \gamma_4\gamma_2}{s_2^2}\gamma_2$$
$$s_5^2 = \alpha_5^2 + \beta_5^2 + \gamma_5^2$$
$$S = \tfrac{1}{2}(s_1 + s_4 + s_3)$$

For control only:

$$s-s_2=\frac{S(S-s_1)(S-s_4)(S-s_3)}{s(s-s_1)(s-s_3)}$$

$$R^2=\frac{S(S-s_1)(S-s_4)(S-s_3)}{s_2^{\,2}} \qquad \tan\tfrac{1}{2}(v_2-v_1)=\frac{R}{s-s_3}$$

$$N=A_1R_1+R_2+A_3R_3 \qquad \tan\tfrac{1}{2}(v_3-v_2)=\frac{R}{s-s_1}$$

$$p=\frac{2R^2s}{N(s-s_1)(s-s_3)} \qquad \tan\tfrac{1}{2}(v_3-v_1)=\frac{Rs}{(s-s_1)(s-s_3)}$$

The computer should be careful to use the corrected values of A_1, A_3. (See VIII.) Trifling errors in the angles should be distributed.

$$e\sin\tfrac{1}{2}(v_3+v_1)=\frac{\frac{p}{r_1}-\frac{p}{r_3}}{2\sin\tfrac{1}{2}(v_3-v_1)}$$

$$e\cos\tfrac{1}{2}(v_3+v_1)=\frac{\frac{p}{r_1}+\frac{p}{r_3}-2}{2\cos\tfrac{1}{2}(v_3-v_1)}$$

$$\tan\tfrac{1}{2}(v_3+v_1) \qquad e^2$$

For control:

$$e\cos v_2=\frac{p}{r_2}-1$$

$$\varepsilon=\sqrt{\frac{1-e}{1+e}} \qquad a=\frac{p}{1-e^2} \qquad b=\sqrt{(ap)}$$

Direction-cosines of semi-major-axis.

$$l=-\frac{\cos v_2}{s_2}\alpha_2-\frac{\sin v_2}{s_5}\alpha_5$$

$$m=-\frac{\cos v_2}{s_2}\beta_2-\frac{\sin v_2}{s_5}\beta_5$$

$$n=-\frac{\cos v_2}{s_2}\gamma_2-\frac{\sin v_2}{s_5}\gamma_5$$

Direction-cosines of semi-minor-axis.

$$\lambda=-\frac{\sin v_2}{s_2}\alpha_2+\frac{\cos v_2}{s_5}\alpha_5$$

$$\mu=-\frac{\sin v_2}{s_2}\beta_2+\frac{\cos v_2}{s_5}\beta_5$$

$$\nu=-\frac{\sin v_2}{s_2}\gamma_2+\frac{\cos v_2}{s_5}\gamma_5$$

Components of the semi-axes.

$$a_x=al \qquad a_y=am \qquad a_z=an$$

$$b_x=b\lambda \qquad b_y=b\mu \qquad b_z=b\nu$$

X.

Time of perihelion passage.

Corrections for aberration.

$$\tan \tfrac{1}{2}E_1 = \varepsilon \tan \tfrac{1}{2}v_1 \qquad \delta t_1 = -.0057613\rho_1$$

$$\tan \tfrac{1}{2}E_2 = \varepsilon \tan \tfrac{1}{2}v_2 \qquad \delta t_2 = -.0057613\rho_2$$

$$\tan \tfrac{1}{2}E_3 = \varepsilon \tan \tfrac{1}{2}v_3 \qquad \delta t_3 = -.0057613\rho_3$$

$$\log .0057613 = 7.76052$$

$$t_1 + \delta t_1 - T = k^{-1}a^{\frac{3}{2}}(E_1 - e \sin E_1)$$

$$t_2 + \delta t_2 - T = k^{-1}a^{\frac{3}{2}}(E_2 - e \sin E_2)$$

$$t_3 + \delta t_3 - T = k^{-1}a^{\frac{3}{2}}(E_3 - e \sin E_3)$$

The threefold determination of T affords a control of the exactness of the solution of the problem. If the discrepancies in the values of T are such as to require another correction of the formulæ (a third hypothesis), this may be based on the equations

$$\Delta \log \tau_1 = M\frac{T_{(3)} - T_{(2)}}{t_3 - t_2} \qquad \Delta \log \tau_3 = M\frac{T_{(2)} - T_{(1)}}{t_2 - t_1}$$

where $T_{(1)}$, $T_{(2)}$, $T_{(3)}$ denote, respectively, the values obtained from the first, second, and third observations, and M the modulus of common logarithms.

XI.

For an ephemeris.

$$\frac{k}{a^{\frac{3}{2}}}(t - T) = E - e \sin E$$

Heliocentric co-ordinates. (Components of $\mathfrak{R}$.)

$$x = -ea_x + a_x \cos E + b_x \sin E$$
$$y = -ea_y + a_y \cos E + b_y \sin E$$
$$z = -ea_z + a_z \cos E + b_z \sin E$$

These equations are completely controlled by the agreement of the computed and observed positions and the following relations between the constants:

$$a_xb_x + a_yb_y + a_zb_z = 0 \qquad a_x^2 + a_y^2 + a_z^2 = a^2 \qquad b_x^2 + b_y^2 + b_z^2 = (1 - e^2)a^2$$

XII.

When an approximate orbit is known in advance, we may use it to improve our fundamental equation. The following appears to be the most simple method:

Find the excentric anomalies E_1, E_2, E_3, and the heliocentric distances r_1, r_2, r_3, which belong in the approximate orbit to the times of observation corrected for aberration.

Calculate B_1, B_3, as in § I, using these corrected times.

Determine A_1, A_3 by the equation

$$\frac{A_1\left(1 + \frac{B_1}{r_1^3}\right)}{\sin(E_3 - E_2) - e \sin E_3 + e \sin E_2} = \frac{A_3\left(1 + \frac{B_3}{r_3^3}\right)}{\sin(E_2 - E_1) - e \sin E_2 + e \sin E_1}$$

in connection with the relation $A_1 + A_3 = 1$.

Determine B_2 so as to make

$$\frac{A_1\frac{B_1}{r_1^3}+\frac{B_2}{r_2^3}+A_3\frac{B_3}{r_3^3}}{4\sin\frac{1}{2}(E_2-E_1)\sin\frac{1}{2}(E_3-E_2)\sin\frac{1}{2}(E_3-E_1)}$$

equal to either member of the last equation.

It is not necessary that the times for which E_1, E_2, E_3, r_1, r_2, r_3, are calculated should precisely agree with the times of observation corrected for aberration. Let the former be represented by t_1', t_2', t_3', and the latter by t_1'', t_2'', t_3''; and let

$$\Delta\log\tau_1=\log(t_3''-t_2'')-\log(t_3'-t_2'),$$

$$\Delta\log\tau_3=\log(t_2''-t_1'')-\log(t_2'-t_1').$$

We may find B_1, B_3, A_1, A_3, B_2, as above, using t_1', t_2', t_3', and then use $\Delta\log\tau_1$, $\Delta\log\tau_2$ to correct their values, as in § VIII.

NUMERICAL EXAMPLE.

To illustrate the numerical computations we have chosen the following example, both on account of the large heliocentric motion, and because Gauss and Oppolzer have treated the same data by their different methods.

The data are taken from the Theoria Motus, § 159, viz:

Times, 1805, September.....	5.51336	139.42711	265.39813
Longitudes of Ceres	95° 32′ 18″.56	99° 49′ 5″.87	118° 5′ 28″.85
Latitudes of Ceres	−0° 59′ 34″.06	+7° 16′ 36″.80	+7° 38′ 49″.39
Longitudes of the Earth.....	342° 54′ 56″.00	117° 12′ 43″.25	241° 58′ 50″.71
Logs of the Sun's distance...	0.0031514	9.9929861	0.0056974

The positions of Ceres have been freed from the effects of parallax and aberration.

I.

From the given times we obtain the following values:

	Numbers.	Logarithms.
t_2-t_1	133.91375	2.1268252
t_3-t_2	125.97102	2.1002706
t_3-t_1	259.88477	2.4147809
A_1	.4847187	9.6854897
A_3	.5152812	9.7120443
τ_1		.3358520
τ_3		.3624066
B_1		9.6692113
B_2		.3183722
B_3		9.5623916

Control:

$$A_1B_1+B_2+A_3B_3=2.4959086$$

$$\tfrac{1}{2}\tau_1\tau_3=2.4959081$$

II.

From the given positions we get:

$\log X_1$	9.9835515	+	$\log X_2$	9.6531725	−	$\log X_3$	9.6775810	−
$\log Y_1$	9.4711748	−	$\log Y_2$	9.9420444	+	$\log Y_3$	9.9515547	−
Z_1	0		Z_2	0		Z_3	0	
$\log \xi_1$	8.9845270	−	$\log \xi_2$	9.2282738	−	$\log \xi_3$	9.6690294	−
$\log \eta_1$	9.9979027	+	$\log \eta_2$	9.9900800	+	$\log \eta_3$	9.9416855	+
$\log \zeta_1$	8.2387150	−	$\log \zeta_2$	9.1026549	+	$\log \zeta_3$	9.1240813	+
$\mathfrak{E}_1\cdot\mathfrak{F}_1$	.3874081	−	$\mathfrak{E}_2\cdot\mathfrak{F}_2$	.9314223	+	$\mathfrak{E}_3\cdot\mathfrak{F}_3$	.5599304	−
p_1^2	.8645336	+	p_2^2	.1006681	+	p_3^2	.7130624	+

III.

The preceding computations furnish the numerical values for the equations III_1, III', III_2, III'', III_3, III''', which follow. Brackets indicate that logarithms have been substituted for numbers.

We have now to assume some values for the heliocentric distances r_1, r_2, r_3. A mean proportional between the mean distances of Mars and Jupiter from the Sun suggests itself as a reasonable assumption. In order, however, to test the convergence of the computations, when the assumptions are not happy, we will make the much less probable assumption (actually much farther from the truth) that the heliocentric distances are an arithmetical mean between the distances of Mars and Jupiter. This gives .526 for the logarithm of each of the distances r_1, r_2, r_3. From these assumed values we compute the first columns of numbers in the three following tables.

$$q_1=\rho_1-.3874081 \qquad r_1^2=q_1^2+.8645336 \qquad R_1=[9.6692113]r_1^{-3}$$

$$\left.\begin{aligned}\alpha_1&=-[8.6700167](q_1-9.5901555)(1+R_1)\\ \beta_1&=\ [9.6833924](q_1+\ .0900552)(1+R_1)\\ \gamma_1&=-[7.9242047](q_1+\ .3874081)(1+R_1)\end{aligned}\right\}III_1$$

$$P'=\frac{[.47712]R_1q_1}{(1+R_1)r_1^2}$$

$$\left.\begin{aligned}\alpha'&=-.046775-[8.67002]R_1-P'\alpha_1\\ \beta'&=\ .482383+[9.68339]R_1-P'\beta_1\\ \gamma'&=-.008399-[7.92420]R_1-P'\gamma_1\end{aligned}\right\}III'$$

Δq_1			− .66731	− .04558	− .0010434	+ .0000006
q_1	+	3.22606	2.55875	2.51317	2.5142134	2.5142140
$\log r_1$	+	.52600	.434960	.4280791	.4282376	.4282377
$\log R_1$	+	8.09121	8.364331	8.3849740	8.3844985	
$\log(1+R_1)$	+	.00533	.009934	.0104122	.0104010	
$\log P'$	+	8.01967	8.369626	8.3957468	8.3951457	
α_1	+	.30136	.336506	.3390605	.3390018	
β_1	+	1.61938	1.307304	1.286223	1.2867056	
γ_1	−	.03072	.025316	.0249518	.0249601	
α'	−	.050505		.0563438		
β'	+	.47139		.4620942		
γ'	−	.00818		.0079821		

$$q_2=\rho_2+.9314223 \qquad r_2^2=q_2^2+.1006681 \qquad R_2=[0.3183722]r_2^{-3}$$

$$\left.\begin{aligned}\alpha_2&=+[9.2282738](q_2+1.7286820)(1-R_2)\\ \beta_2&=-[9.9900800](q_2-\ .0361309)(1-R_2)\\ \gamma_2&=-[9.1026549](q_2-\ .9314223)(1-R_2)\end{aligned}\right\}III_2$$

$$P''=\frac{[.47712]R_2q_2}{(1-R_2)r_2^2}$$

$$\left.\begin{aligned}\alpha''&=\ .169151-[9.22827]R_2+P''\alpha_2\\ \beta''&=-.977417+[9.99008]R_2+P''\beta_2\\ \gamma''&=-.126664+[9.10265]R_2+P''\gamma_2\end{aligned}\right\}III''$$

Δq_2			− .77826	+ .005042	+ .0013222	+ .0000021
q_2	+	3.34235	2.56409	2.569132	2.5704542	2.5704563
$\log r_2$	+	.52600	.412233	.4130733	.4132934	.4132937
$\log R_2$	+	8.74037	9.081673	9.0791524	9.0784920	
$\log (1-R_2)$	+	9.97543	9.944142	9.9444866	9.9445766	
$\log P''$	+	8.71411	9.199120	9.1954270	9.1944598	
α_2	+	.81059	.638489	.6397466	.6400760	
β_2	−	3.05379	2.172660	2.1787230	2.1803116	
γ_2	−	.28858	.181843	.1825486	.1827338	
α''	+	.20182		.2491854		
β''	−	1.08177		1.2018221		
γ''	−	.13464		.1400944		

$$q_3=\rho_3-.5599304 \qquad r_3^2=q_3^2+.7130624 \qquad R_3=[9.5623916]r_3^{-3}$$

$$\left.\begin{aligned}\alpha_3&=-[9.3810737]\,(q_3+1.5798163)\,(1+R_3)\\ \beta_3&=[9.6537308]\,(q_3-\ .4630521)\,(1+R_3)\\ \gamma_3&=[8.8361256]\,(q_3+\ .5599304)\,(1+R_3)\end{aligned}\right\}\mathrm{III}_3$$

$$P'''=\frac{[.47712]R_3q_3}{(1+R_3)r_3^2}$$

$$\left.\begin{aligned}\alpha'''&=-.240477-[9.38107]R_3-P'''\alpha_3\\ \beta'''&=+.450537+[9.65373]R_3-P'''\beta_3\\ \gamma'''&=+.068569+[8.83613]R_3-P'''\gamma_3\end{aligned}\right\}\mathrm{III}'''$$

Δq_3			− .80780	− .04055	+ .0025316	+ .0000031
q_3	+	3.24945	2.44165	2.40110	2.4036316	2.4036347
$\log r_3$	+	0.52600	.412217	.4057319	.4061394	.4061399
$\log R_3$	+	7.98439	8.325742	8.3451948	8.3439733	
$\log (1+R_3)$	+	.00417	.009099	.0095108	.0094843	
$\log P'''$	+	7.91715	8.357016	8.3817516	8.3801993	
α_3	−	1.17253	.987590	.9785152	.9790776	
β_3	+	1.26749	.910305	.8924956	.8936069	
γ_3	+	.26373	.210171	.2075292	.2076940	
α'''	−	.22847		.2222335		
β'''	+	.44441		.4390163		
γ'''	+	.06690		.0650888		

IV.

The values of α', β', etc., furnish the basis for the computation of the following quantities:

$$a_1=-.01254 \qquad a_2=-.03517 \qquad a_3=-.07232$$
$$b_1=+.01726 \qquad b_2=-.00525 \qquad b_3=-.00845$$
$$c_1=-.15746 \qquad c_2=-.08526 \qquad c_3=-.04050$$

For G we get three values sensibly identical. Adopting the mean, we set

$$G=.01006.$$

We also get

$$H=-.00998, \qquad L=.02322.^*$$

V.

Taking the values of α_1, α_2, etc., from the columns under III_1, III_2, III_3, we form the residuals

$$\alpha=-.06058, \qquad \beta=-.16692, \qquad \gamma=-.05557.$$

From these, with the numbers last computed, we get

$$C_1=-.65888, \qquad C_2=-.76983, \qquad C_3=-.79939,$$

* It would have been better to omit altogether the calculation of H and L, if the small value of the latter could have been foreseen. In fact, it will be found that the terms containing L hardly improve the convergence, being smaller than quantities which have been neglected. Nevertheless, the use of these terms in this example will illustrate a process which in other cases may be beneficial.

which might be used as corrrections for our values of q_1, q_2, q_3. To get more accurate values for these corrections we set

$$\Delta q_2 = C_2 - \tfrac{6}{10} L(\Delta q_2)^2, \qquad \text{or } \Delta q_2 = -.76983 - .01393(\Delta q_2)^2,$$

which gives

$$\Delta q_2 = -.77826.$$

The quadratic term diminishes the value of Δq_2 by .00843. Subtracting the same quantity from C_1 and C_2 we get

$$\Delta q_1 = -.66731, \qquad \Delta q_3 = -.80780.$$

VI.

Applying these corrections to the values of q_1, q_2, q_3 we compute the second numerical columns under equations III_1, III_2, and III_3. We do not go on to the computations by equations III′, etc., but content ourselves with the old values of a_1, b_1, etc., G, and L, which with the new residuals,

$$\alpha = -.012595, \qquad \beta = .044949, \qquad \gamma = .003012,$$

give

$$C_1 = -.04567, \qquad C_2 = .004952, \qquad C_3 = -.04064.$$

$$\Delta q_2 = C_2 - L(Dq_2 + \tfrac{1}{2}C_2)\Delta q_2 = .004952 - .02322(-.77826 + .00247)\Delta q_2.$$

This gives

$$\Delta q_2 = .005042$$

As the term containing L has increased the value of Δq_2 by .00009, we add this quantity to C_1 and C_3, and get

$$\Delta q_1 = -.04558, \qquad \Delta q_3 = -.04055.$$

With these corrections we compute the third numerical columns under equations III_1, etc. This time we recompute the quantities α', etc., with which we repeat the principal computations of IV, and get the new values:

$$a_1 = -.0167215 \qquad a_2 = -.0335815 \qquad a_3 = -.0743299$$

$$b_1 = +.0149145 \qquad b_2 = -.0054413 \qquad b_3 = -.0098825$$

$$c_1 = -.1576886 \qquad c_2 = -.0779570 \qquad c_3 = -.0474318$$

$$G = .0090929$$

The quantities H and L we neglect as of no consequence at this stage of the approximation.

With these values the new residuals,

$$\alpha = +.0002919, \qquad \beta = -.0000044, \qquad \gamma = +.0000288,$$

give

$$\Delta q_1 = C_1 = +.0010434, \qquad \Delta q_2 = C_2 = +.0013222, \qquad \Delta q_3 = C_3 = +.0025316.$$

These corrections furnish the basis for the fourth columns of numbers under equations III_1, etc., which give the residuals

$$\alpha = +.0000002, \qquad \beta = +.0000009, \qquad \gamma = +.0000001,$$

and the new corrections

$$\Delta q_1 = +.0000006, \qquad \Delta q_2 = +.0000021, \qquad \Delta q_3 = +.0000031.$$

The corrected values of q_1, q_2, q_3 give

$$\log r_1 = 0.4282377, \qquad \log r_2 = 0.4132937, \qquad \log r_3 = 0.4061399.$$

We have carried the approximation farther than is necessary for the following *correction of the formula,* in order to see exactly where the uncorrected formula would lead us, and for the control afforded by the fourth residuals.

VII.

The computations for the test of the uncorrected formula (the first hypothesis) are as follows:

		Number or arc.	Logarithm.			Number or arc.	Logarithm.
r_1			0.4282377	e	+		8.9025438
r_2			0.4132937	ε	+		9.9652259
r_3			0.4061399	a	+		0.4419546
$A_1B_1r_1^{-3}$	+	.01174865	8.0699879	$\tan \frac{1}{2}v_1$	—	−35° 41′ 39″.75	9.8563809
$B_2r_2^{-3}$	+	.11980944	9.0784911	$\tan \frac{1}{2}v_2$	—	−19° 53′ 28″.93	9.5584981
$A_3B_3r_3^{-3}$	+	.01137670	8.0560162	$\tan \frac{1}{2}v_3$	—	− 4° 13′ 52″.55	8.8691380
N	+	.14293479	9.1551380	$\tan \frac{1}{2}E_1$	—	−33° 33′ 0″.17	9.8216068
s_1	+	1.3308476	0.1241283	$\tan \frac{1}{2}E_2$	—	−18° 28′ 6″.35	9.5237240
s_2	+	2.2796616	0.3578704	$\tan \frac{1}{2}E_3$	—	− 3° 54′ 24″.21	8.8343639
s_3	+	1.3417404	0.1276685	$\sin E_1$	—	−67° 6′ 0″.34	9.9643473
s	+	2.4761248	0.3937725	$\sin E_2$	—	−36° 56′ 12″.70	9.7788272
$s-s_1$	+	1.1452772	0.0589106	$\sin E_3$	—	− 7° 48′ 48′.42	9.1333734
$s-s_2$	+	0.1964632	9.2932812	$ea^{\frac{3}{2}} \sin E_1$	—	.3387061	9.5298230
$s-s_3$	+	1.1343844	0.0547602	$ea^{\frac{3}{2}} \sin E_2$	—	.2209545	9.3443029
R	+		9.5065898	$ea^{\frac{3}{2}} \sin E_3$	—	.0499861	8.6988491
p	+		0.4391732	$a^{\frac{3}{2}} (E_2-E_1)$	+	2.4226307	0.3842872
$\tan \frac{1}{2}(v_2-v_1)$	+	15° 48′ 10″.82	9.4518296	$a^{\frac{3}{2}} (E_3-E_2)$	+	2.3391145	0.3690515
$\tan \frac{1}{2}(v_3-v_2)$	+	15° 39′ 36″.38	9.4476792	τ_3 calc.	+	2.3048791	0.3626482
$\tan \frac{1}{2}(v_3-v_1)$	+	31° 27′ 47″.20	9.7866915	τ_1 calc.	+	2.1681461	0.3360885
$e \sin \frac{1}{2}(v_3+v_1)$	—		8.7099387				
$e \cos \frac{1}{2}(v_3+v_1)$	+		8.7872701				
$\tan \frac{1}{2}(v_3+v_1)$	—	−39° 55′ 32″.31	9.9226686				

VIII.

The logarithms of the calculated values of the intervals of time exceed those of the given values by .0002416 for the first interval (τ_3) and .0002365 for the second (τ_1). Therefore, since the corrections for aberration have been incorporated in the data, we set for the correction of the formula (for the second hypothesis)

$$\Delta \log \tau_1 = -.0002365 \qquad \Delta \log \tau_3 = -.0002416$$

This gives

$$\Delta \log A_1 = .0000026 \qquad \Delta \log A_3 = -.0000025$$

$$\Delta \log B_1 = -.0004872 \qquad \Delta \log B_2 = -.0004782 \qquad \Delta \log B_3 = -.0004665$$

The new values of the logarithms of A_1, A_3 are

$$\log A_1 = 9.6854923 \qquad \log A_3 = 9.7120418$$

The equations for an ephemeris will then be:

$$T=1806,\ \text{June } 23.96378,\ \text{Paris mean time}$$

$$[2.8863140](t-T)=E_{\text{in seconds}}-[4.2216530]\sin E$$

Heliocentric coördinates relating to the ecliptic.

$$x=+.1820765-[0.3530261]\cos E-[0.1827783]\sin E$$

$$y=-.1244853+[0.1878904]\cos E-[0.3603153]\sin E$$

$$z=-.0373987+[9.6656285]\cos E+[9.3320758]\sin E$$

The agreement of the calculated geocentric positions with the data is shown in the following table:

Times, 1805, September	5.51336	139.42711	265.39813
Second hypothesis:			
longitudes	95°32′18″.88	99°49′ 5″.87	118° 5′28″.52
errors	0″.32	0″.00	−0″.33
latitudes	−0°59′34″.01	7°16′36″.82	7°38′49″.34
errors	0″.05	0″.02	−0″.05
Third hypothesis:			
longitudes	95°32′18″.65	99°49′ 5″.82	118° 5′28″.79
errors	0″.09	−0″.05	−0″.06
latitudes	−0°59′34″.04	7°16′36″.78	7°38′49″.38
errors	0″.02	−0″.02	−0″.01

The immediate result of each hypothesis is to give three positions of the planet, from which, with the times, the orbit may be calculated in various ways, and with different results, so far as the positions deviate from the truth on account of the approximate nature of the hypothesis. In some respects, therefore, the correctness of an hypothesis is best shown by the values of the geocentric or heliocentric distances which are derived directly from it. The logarithms of the heliocentric distances are brought together in the following table, and corresponding values from Gauss* and Oppolzer† are added for comparison. It is worthy of notice that the positions given by our second hypothesis are substantially correct, and if the orbit had been calculated from the first and third of these positions with the interval of time, it would have left little to be desired.

	$\log r_1$.	$\log r_2$.	$\log r_3$.
First hypothesis	.4282377	.4132937	.4061399
Second hypothesis	.4282782	.4132809	.4061998
Third hypothesis	.4282786	.4132808	.4062003
Gauss:			
First hypothesis	.4323934	.4114726	.4094712
Second hypothesis	.4291773	.4129371	.4071975
Third hypothesis	.4284841	.4132107	.4064697
Fourth hypothesis	.4282792	.4132817	.4062033
Oppolzer:			
First hypothesis	.4281340	.413330	.4061699
Second hypothesis	.4282794	.4132801	.4061976
Third hypothesis	.4282787		.4062009

In comparing the different methods, it should be observed that the determination of the positions in any hypothesis by Gauss's method requires successive corrections of a single independent variable, a corresponding determination by Oppolzer's method requires the successive corrections of two independent variables, while the corresponding determination by the method of the present paper requires the successive corrections of three independent variables.

* Theoria motus, § 159. † Lehrbuch zur Bahnbestimmung der Kometen und Planeten, 2d ed., vol. I, p. 394.